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HOMOGENEOUS SOLUTIONS OF THE DIRICHLET PROBLEM FOR AN ANISOTROPIC LAYER†

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A complete system of homogeneous solutions of the Dirichlet problem for an anisotropic layer is constructed. These solutions represent series containing metaharmonic functions of a complex argument which depends on all three coordinates. The solution obtained can be used when considering boundary-value problems of potential theory for a piecewise-homogeneous layer.

Many problems in hydrodynamics, steady heat conduction, electro- and magnetostatics for essentially anisotropic media, reduce to the integration of a general second-order differential equation of elliptic type. When the corresponding boundary-value problems are being considered for a piecewise-homogeneous layer, it is desirable to use the method of homogeneous solutions. This method, which goes back to the work of Lur'ye [1], has proved extremely effective in investigating the stress-strain of isotropic or transversely isotropic thick plates [2]. However, when bodies with anisotropy of a general type are considered, difficulties arise associated with the construction of complete systems of homogeneous solutions of the boundary-value problems. In order to demonstrate what is involved and point out analytical procedures for constructing homogeneous solutions, a homogeneous Dirichlet problem will be considered for an anisotropic layer.

1. FORMULATION OF THE PROBLEM. THE OPERATOR APPROACH

We are concerned with integrating a differential equation of elliptic type with constant coefficients

$$\mathbf{L(D)u} = 0$$
$$\mathbf{L(D)} = \sum_{m,n=1}^{3} a_{mn}\partial_m\partial_n, \quad \partial_n = \frac{\partial}{\partial x_n}, \quad a_{nm} = a_{mn}, \quad a_{33} > 0 \tag{1.1}$$

in a layer $-\infty < x_1, x_2 < \infty$, $-h < x_3 < h$, satisfying homogeneous boundary conditions on the bases of the layer

$$u|_{x_3 = \pm h} = 0 \tag{1.2}$$

and also the conditions for the solution to attenuate at infinity ($|x_1| \to \infty$, $|x_2| \to \infty$).

The boundary-value problem (1.1), (1.2) will be solved by the operator method. Setting $u' = \partial_3 u$, $u'' = \partial_3^2 u$, we represent Eq. (1.1) in the form

$$u'' + 2Au' + Bu = 0$$

$$A = \frac{1}{a_{33}}(a_{13}\partial_1 + a_{23}\partial_2), \quad B = \frac{1}{a_{33}}(a_{11}\partial_1^2 + 2a_{12}\partial_1\partial_2 + a_{22}\partial_2^2)$$

Integration of this equation, taking the ellipticity of the operator $L(D)$ into consideration, gives

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$$u = e^{-Ax_3}\{\cos\alpha x_3 C_1 + \alpha^{-1}\sin\alpha x_3 C_2\}$$

$$\alpha^2 = B - A^2 = \Delta_{11}\partial_1^2 + 2\Delta_{12}\partial_1\partial_2 + \Delta_{22}\partial_2^2$$

$$\Delta_{11} = \frac{a_{11}}{a_{33}} - \left(\frac{a_{13}}{a_{33}}\right)^2, \quad \Delta_{12} = \frac{a_{12}}{a_{33}} - \frac{a_{13}a_{23}}{a_{33}^2} \tag{1.3}$$

$$\Delta_{22} = \frac{a_{22}}{a_{33}} - \left(\frac{a_{23}}{a_{33}}\right)^2 > 0$$

$$C_1 = C_1(x_1, x_2), \quad C_2 = C_2(x_1, x_2)$$

To determine the functions $C_k(x_1, x_2)$ $(k = 1, 2)$ we invoke boundary conditions (1.2). Taking expression (1.3) into consideration, we obtain a system of operator equations

$$(\cos h\alpha \operatorname{ch} hA)C_1 - (\alpha^{-1}\sin h\alpha \operatorname{sh} hA)C_2 = 0$$
$$(\cos h\alpha \operatorname{sh} hA)C_1 - (\alpha^{-1}\sin h\alpha \operatorname{ch} hA)C_2 = 0 \tag{1.4}$$

We introduce the resolvent ψ by relations

$$C_1 = (\alpha^{-1}\sin h\alpha \operatorname{sh} hA)\psi, \quad C_2 = (\cos h\alpha \operatorname{ch} hA)\psi \tag{1.5}$$

Then the first equation of system (1.4) will be satisfied, while the second reduces to the following equation for the function $\psi = \psi(x_1, x_2)$

$$(\alpha^{-1}\sin 2h\alpha)\psi = 0 \tag{1.6}$$

Let us express the operator function occurring in Eq. (1.6) as an operator series. We obtain an equation of infinite order in ψ

$$\left(\sum_{k=0}^{\infty}(-1)^k\frac{(2h)^{2k+1}}{(2k+1)!}\alpha^{2k}\right)\psi = 0 \tag{1.7}$$

To solve it, we introduce a system of functions $\varphi_j(x_1, x_2)$ satisfying the equation

$$(\alpha^2 - \mu_j^2)\varphi_j = 0, \tag{1.8}$$

where μ_j are as yet unknown parameters.

It follows from Eq. (1.8) that $\alpha^{2k}\varphi_j = \mu_j^{2k}\varphi_j$, and Eq. (1.7), as applied to the function φ_j, gives

$$\frac{1}{\mu_j}\sin(2h\mu_j)\varphi_j = 0$$

Thus, non-trivial solutions of Eq. (1.7) exist, and the corresponding characteristic numbers μ_j are defined by the equality

$$2h\mu_j = \pi j, \quad j = \pm 1, \pm 2, \pm, \ldots$$

We now require that the solution of Eq. (1.1) should attenuate as $r = \sqrt{x_1^2 + x_2^2} \to \infty$. This implies that μ_j is positive, and we can therefore write

$$\psi = \sum_{j=1}^{\infty}\varphi_j(x_1, x_2) \tag{1.9}$$

We will now determine the eigenfunctions φ_j $(j = 1, 2, \ldots)$.

Introducing a non-singular coordinate transformation

$$x_1^* = x_1 - px_2, \quad x_2^* = \sqrt{\Delta^*}x_2$$
$$p = \frac{\Delta_{12}}{\Delta_{22}}, \quad q = \frac{\Delta_{11}}{\Delta_{22}}, \quad \Delta^* = q - p^2 > 0 \tag{1.10}$$

we reduce Eq. (1.8) to the form

$$(\nabla_*^2 - \gamma_j^2)\varphi_j = 0, \quad j = 1, 2, \ldots$$
$$\nabla_*^2 = \partial_{1*}^2 + \partial_{2*}^2 = \frac{\alpha^2}{\Delta^*\Delta_{22}} \tag{1.11}$$
$$\gamma_j^2 = \frac{\mu_j^2}{\Delta^*\Delta_{22}}; \quad \partial_{1*} = \frac{\partial}{\partial x_1^*}, \quad \partial_{2*} = \frac{\partial}{\partial x_2^*}$$

Hence it follows that φ_j are metaharmonic functions in the affine variables x_1^*, x_2^*.

We will now consider the construction of a solution of the original boundary-value problem. Substituting the functions C_1, C_2 (1.5) into Eq. (1.3) we obtain

$$u = \left\{\frac{1}{2}e^{A(h-x_3)}\alpha^{-1}\sin[\alpha(h+x_3)] - \frac{1}{2}e^{-A(h+x_3)}\alpha^{-1}\sin[\alpha(h-x_3)]\right\}\psi$$

Introducing expression (1.9) for the function ψ, we obtain a solution of the original equation (1.1) in the symbolic form

$$u = u_+ - u_-, \quad u_\pm = \frac{1}{2}e^{\pm A(h\mp x_3)}\sum_{j=1}^{\infty}\frac{\varphi_j}{\mu_j}\sin[(h \pm x_3)\mu_j] \tag{1.12}$$

We will now consider the second solution of system (1.4), setting

$$C_1 = (\alpha^{-1}\sin\alpha h\cos Ah)\psi_*, \quad C_2 = (\cos\alpha h\,\mathrm{sh}Ah)\psi_* \tag{1.13}$$

In this case, the second equation of (1.4) is satisfied, while the first leads to the resolvent (1.6). Reasoning as before, we arrive at the solution

$$u_* = u_+ + u_- \tag{1.14}$$

Comparing representations (1.12) and (1.14), we obtain two types of solution of the original boundary-value problem (1.1), (1.2)

$$u_1 = u_+, \quad u_2 = u_-; \quad \mu_j = \pi j/(2h) \tag{1.15}$$

In the special case of a transversely isotropic medium, the functions (1.12) and (1.14) define solutions which are skew-symmetric and symmetric, respectively, with respect to the middle surface of the solution layer.

We will now obtain the result of applying the exponential operator-function to a cylindrical function. To do this we introduce a complex variable $z_* = x_1^* + ix_2^* = r_*e^{i\alpha_*}$ and complex differentiation operators

$$\frac{\partial}{\partial z_*} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^*} - i\frac{\partial}{\partial x_2^*}\right), \quad \frac{\partial}{\partial \bar{z}_*} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^*} + i\frac{\partial}{\partial x_2^*}\right)$$

In these variables the form of the operator A will be

$$A = \beta_*\frac{\partial}{\partial z_*} + \bar{\beta}_*\frac{\partial}{\partial \bar{z}_*}; \quad \beta_* = \frac{a_{13} + \nu_* a_{23}}{a_{33}} = |\beta_*|e^{i\delta}, \quad \nu_* = i\sqrt{\Delta^*} - p \tag{1.16}$$

Since an arbitrary solution of Eq. (1.11) is the convolution of a simple or double layer with the MacDonald function $K_0(\gamma_j \mathbf{r}^*)$ [3], it will suffice to consider the application of the operator to this function.

We shall prove the validity of the following equalities

$$e^{\pm A(h\mp x_3)}K_0(\gamma_j r_*) = K_0(\gamma_j R_\pm)$$
$$R_\pm = \sqrt{r_*^2 + (h\mp x_3)^2|\beta_*|^2 \pm 2r_*|\beta_*|(h\mp x_3)\cos(\alpha_* - \delta)} = \sqrt{W_\pm\bar{W}_\pm} = |W_\pm| \tag{1.17}$$
$$W_\pm = z_* \pm \beta_*(h \mp x_3)$$

where $\overline{W}$ is the complex conjugate of W.

Taking note of the notation (1.16), we have

$$e^{A(h-x_3)}K_0(\gamma_j r_*) = \sum_{k=0}^{\infty}\frac{(h-x_3)^k}{k!}\left(\beta_*\frac{\partial}{\partial z_*}+\bar{\beta}_*\frac{\partial}{\partial \bar{z}_*}\right)^k K_0(\gamma_j r_*) \tag{1.18}$$

Using the easily proved relations [4]

$$\frac{\partial^n}{\partial z^n}K_0(\gamma r) = \left(-\frac{\gamma}{2}\right)^n e^{-in\alpha}K_n(\gamma r)$$

$$\frac{\partial^n}{\partial \bar{z}^n}K_0(\gamma r) = \left(-\frac{\gamma}{2}\right)^n e^{in\alpha}K_n(\gamma r)$$

$$z = x_1 + ix_2 = re^{i\alpha}, \quad n = 0, 1, \ldots$$

we obtain, after some reduction,

$$\begin{gathered}\left(\beta_*\frac{\partial}{\partial z_*}+\bar{\beta}_*\frac{\partial}{\partial \bar{z}_*}\right)^{2k} K_0(\gamma_j r_*) = \\ = 2\left(\frac{|\beta_*|\gamma_j}{2}\right)^{2k}\sum_{m=0}^{k}\delta_m C_{2k}^{k-m}\cos[2m(\alpha_*-\delta)]K_{2m}(\gamma_j r_*) \\ \left(\beta_*\frac{\partial}{\partial z_*}+\bar{\beta}_*\frac{\partial}{\partial \bar{z}_*}\right)^{2k+1} K_0(\gamma_j r_*) = \\ = -2\left(\frac{|\beta_*|\gamma_j}{2}\right)^{2k+1}\sum_{m=0}^{k} C_{2k+1}^{k-m}\cos[(2m+1)(\alpha_*-\delta)]K_{2m+1}(\gamma_j r_*) \\ k = 0, 1, \ldots \\ \delta_m = \begin{cases} 1/2, & m = 0 \\ 1, & m = 1, 2, \ldots \end{cases}, \quad C_m^n = \frac{m!}{n!(m-n)!}\end{gathered} \tag{1.19}$$

Substituting expressions (1.19) into (1.18) we obtain

$$e^{A(h-x_3)}K_0(\gamma_j r_*) = X_1 - X_2$$

$$X_1 = 2\sum_{m=0}^{\infty}\delta_m\cos[2m(\alpha_*-\delta)]I_{2m}[\gamma_j|\beta_*|(h-x_3)]K_{2m}(\gamma_j r_*)$$

$$X_2 = 2\sum_{m=0}^{\infty}\cos[(2m+1)(\alpha_*-\delta)]I_{2m+1}[\gamma_j|\beta_*|(h-x_3)]K_{2m+1}(\gamma_j r_*)$$

where $K_n(z), I_n(z)$ are the MacDonald function and the modified Bessel function, respectively, of order n.

Finally, using the Graf Addition Theorem [5], we obtain equality (1.17) with the plus sign chosen. Replacing h by $-h$ in that equality, we obtain (1.17) with the minus sign chosen. Incidentally, relations (1.17) are of independent interest in the theory of cylindrical functions.

Thus, using formulae (1.17), we obtain a coordinate realization of the operator equalities (1.15) in the form

$$u_1 = u_+, \quad u_2 = u_-$$

$$u_\pm = \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{\mu_j}K_0(\gamma_j R_\pm)\sin[(h\pm x_3)\mu_j] \tag{1.20}$$

The functions $u_n = u_n(x_1, x_2, x_3)$ (1.20) have discontinuities of the second kind on the curves $W_\pm = 0$. Taking the expressions (1.17) into account for the complex variable $W_\pm$, we obtain a pair of straight lines

$$x_1 = \frac{a_{13}}{a_{33}}(x_3 \pm h), \quad x_2 = \frac{a_{23}}{a_{33}}(x_3 \pm h)$$

where the upper sign corresponds to the function u_2 and the lower one to u_1. As one might expect, in the case $a_{13} = a_{23} = 0$ functions (1.20) have discontinuities in the interval $-h < x_3 < h$ of the x_3 axis.

2. THE GENERAL FORM OF HOMOGENEOUS SOLUTIONS OF BOUNDARY-VALUE PROBLEM (1.1), (1.2)

It is obvious from the construction of the homogeneous solutions (1.20) that the function $K_0(\gamma_j R_+)$ is a solution of the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial y_*^2} - \gamma_j^2\right)\Phi(\gamma_j|W_+|) = 0 \tag{2.1}$$

where

$$x_* = \mathrm{Re}\,W_+, \quad y_* = \mathrm{Im}\,W_+, \quad |W_+| = R_+ \tag{2.2}$$

Similarly, the function $K_0(\gamma_j R_-)$ is a solution of the equation with the subscript plus replaced by minus in (2.2).

We shall show that expressions (1.20) yield solutions of boundary-value problem (1.1), (1.2) if the MacDonald function K_0 in them is replaced by an arbitrary sufficiently smooth solution of Eq. (2.1). To do this, we write the function u_1 in the form

$$u_1 = \sum_{j=1}^{\infty} \frac{1}{\mu_j}\Phi_j^+ \sin[(h + x_3)\mu_j] \tag{2.3}$$

The quantities μ_j were defined above, and the functions $\Phi_j^+ = \Phi_j^+(x_*, y_*)$ are solutions of Eq. (2.1) in the case (2.2).

Substituting expression (2.3) into Eq. (1.1) we obtain the system

$$\tilde{A}\Phi_j^+ = 0, \quad \tilde{B}\Phi_j^+ + 2\tilde{A}\partial_3\Phi_j^+ = 0 \tag{2.4}$$

$$\tilde{A} = a_{13}\partial_1 + a_{23}\partial_2 + a_{33}\partial_3$$

$$\tilde{B} = a_{11}\partial_1^2 + 2a_{12}\partial_1\partial_2 + a_{22}\partial_2^2 - a_{33}\partial_3^2 - a_{33}\mu_j^2$$

The first equation of this system is satisfied by any continuously differentiable function $\Phi(x_*, y_*)$. Indeed, in the variables W_+, $\overline{W}_+$, defined by the last equalities of (1.17) it becomes

$$\mathrm{Re}\left\{(a_{13} + a_{23}\nu_* - a_{33}\beta_*)\frac{\partial\Phi_j^+}{\partial W_+}\right\} = 0 \tag{2.5}$$

and this is an identity by virtue of relations (1.16).

The second equation of system (2.4) may be written, using equality (2.5), as

$$2\mathrm{Re}\left(a\frac{\partial^2\Phi_j^+}{\partial W_+^2}\right) + 2b\frac{\partial^2\Phi_j^+}{\partial W_+\partial\overline{W}_+} - a_{33}\mu_j^2\Phi_j^+ = 0 \tag{2.6}$$

$$a = a_{11} + 2a_{12}\nu_* + a_{22}\nu_*^2 - 2\beta_*(a_{13} + a_{23}\nu_*) + a_{33}\beta_*^2$$

$$b = a_{11} + a_{12}|\nu_*|^2 + a_{33}|\beta_*|^2 + 2\mathrm{Re}(a_{12}\nu_* - a_{13}\beta_* - a_{23}\nu_*\overline{\beta}_*)$$

Invoking now the formulae for β_* and ν_* from (1.16), we find

$$a = 0, \quad b = 2a_{33}\Delta^*\Delta_{22}$$

Consequently, Eq. (2.6) is identical with (2.1), as required.

A second solution u_2 may be constructed in the same way as solution (2.3), differing from (2.3) in that x_3 is replaced by $-x_3$ and Φ_j^+ by $\Phi_j^- = \Phi_j^-(x_*, y_*)$ – the solution of Eq. (2.1) in the case when the subscript plus in (2.2) is replaced by minus.

The essential difference between the homogeneous equations obtained here and the known analogous solutions, which correspond to isotropic or transversely isotropic media, is that they cannot be constructed by separation of variables in Cartesian coordinates x_k ($k = 1, 2, 3$). This follows from the fact that the function $\Phi_j^{\mp}$ depends on all three variables x_1, x_2, x_3.

3. DISCUSSION OF THE RESULTS

In cases when $|a_{13}|/a_{33} \ll 1$, $|a_{23}|/a_{33} \ll 1$, one can use an approximate procedure to construct homogeneous solutions. Retaining only the zeroth and first powers of the operator A in the first relation of (1.12), we can write

$$e^{A(h-x_3)}\varphi_j = \varphi_j + \frac{h-x_3}{a_{33}}(a_{13}\partial_1 + a_{23}\partial_2)\varphi_j = f_j$$

Hence, and from the first relation of (1.15), we find that

$$u_1 = \sum_{j=1}^{\infty} \frac{1}{\mu_j} \sin[(h+x_3)\mu_j] f_j \tag{3.1}$$

Similarly, one obtains a second solution, u_2, differing from (3.1) in that x_3 is replaced by $-x_3$ and a_{33} by $-a_{33}$.

In the general case, one can find solutions which are symmetric and skew-symmetric about the origin by combining the functions u_1 and u_2. It is obvious from equalities (1.17) that

$$W_+(-x_1, -x_2, -x_3) = -W_-(x_1, x_2, x_3)$$

Applying the following substitution to formula (2.3) and the analogous formula for u_2,

$$\Phi_j^-(x_1, x_2, x_3) = -\Phi_j^+(-x_1, -x_2, -x_3)$$

we obtain the corresponding solutions

$$u^+ = u_1 + u_2, \quad u^- = u_1 - u_2$$

Expression (3.1) for u_1 and the analogous expression for u_2 may be used when solving boundary-value problems for a piecewise-homogeneous medium, such as a layer with continuous tunnel cavities or slits. When that is done, the corresponding problems reduce to systems of homogeneous singular integral equations. When that method is adopted, however, problems associated with the existence of small parameters for singular operators are overlooked. From that point of view, it is preferable to start with the exact expressions (1.20), introducing convolutions of MacDonald functions with a double layer on the surface of inhomogeneity. The integral representations thus obtained for the solutions of boundary-value problems serve as the starting point for reducing the latter to two-dimensional integral equations. The effectiveness of analytical and numerical procedures remains to be evaluated.

REFERENCES

1. LUR'YE, A. I., The theory of thick slabs. *Prikl. Mat. Mekh.*, 1942, **6**, 2/3, 151–168.
2. GRIGOLYUK, E. I. and KOGAN, F. A., The present state of the theory of multilayer shells. *Prikl. Mekhanika*, 1972, **8**, 6, 3–17.
3. VLADIMIROV, V. S., *The Equations of Mathematical Physics*. Nauka, Moscow, 1981.
4. FIL'SHTINSKII, L. A., Interaction of stress waves with curvilinear tunnel cracks of longitudinal shear in a half-space. *Prikl. Mat. Mekh.*, 1982, **46**, 3, 482–487.
5. LUKE, Y. L., *Mathematical Functions and their Approximations*. Academic Press, New York, 1975.

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